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A Solution of Weyl-Lanczos Equations for Arbitrary Petrov Type D Vacuum Spacetimes

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Abstract A Lanczos potential for an arbitrary Petrov type D vacuum spacetimes, using the compacted spin coefficient formalism (or GHP-formalism), has been obtained; which in turn leads to a solution of Weyl-Lanczos equations.

Keywords Petrov type D fields · Lanczos potential · Weyl-Lanczos equation

1 Introduction

Let M be a four dimensional spacetime endowed with a metric g_{ij} of signature $(---+)$. The curvature tensor R^h_{ijk} is defined through the Ricci identity

$$A_{i;j;k} - A_{i;k;j} = R^h_{ijk} A_h \tag{1}$$

for any vector field A_h. The Riemann curvature tensor R_{hijk} can be decomposed as [3]

$$R_{hijk} = C_{hijk} + E_{hijk} + G_{hijk} \tag{2}$$

where

$$E_{hijk} = \frac{1}{2}(g_{hj}S_{ik} + g_{ik}S_{hj} - g_{hk}S_{ij} - g_{ij}S_{hk}) \tag{3}$$

$$G_{hijk} = \frac{R}{12}(g_{hj}g_{ik} - g_{hk}g_{ij}) \tag{4}$$

$$S_{ij} = R_{ij} - \frac{1}{4}g_{ij}R \tag{5}$$

$$R_{ij} = R^k_{ijk}, \qquad R = g^{ij}R_{ij} \tag{6}$$

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The irreducible tensor C_{hijk} is the Weyl tensor and satisfies the same algebraic properties as that of the Riemann curvature tensor and is completely trace-less.

When the Einstein's field equations

$$R_{ij} = 0 \tag{7}$$

are imposed then from (2) all that remains of the gravitational field is the Weyl tensor and it (Weyl tensor) describes the pure gravitational field. However, when gravitational waves propagate through matter, the Weyl tensor is still pertinent.

Moreover, the correspondence electromagnetism and gravitation are very rich and detailed. Some of these correspondence are still uncovered while some of them are further developed. This correspondence is reflected in the Maxwell-like form of the gravitational field tensor (the Weyl tensor), the super energy-momentum tensor (the Bel-Robinson tensor) and the dynamical equations (the Bianchi identities).

Let the spacetime M admits an electromagnetic field F_{ij} such that $F_{ij} = -F_{ji}$ and satisfies the Maxwell's equation

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0 \tag{8}$$

$$F^{ij}_{;j} = 0 \tag{9}$$

Since (8) guarantee the local existence of a 4-potential A_i such that

$$F_{ij} = A_{i;j} - A_{j;i} \tag{10}$$

we can replace (8) by (10). Equation (10) suggests that an electromagnetic field F_{ij} can be generated through a potential A_i.

Now the question is: whether is it possible or not to generate the gravitational field through a potential? The answer is in affirmative. This can be done with the covariant differentiation of a tensor field.

Moreover, since the tensors E_{hijk} and G_{hijk} are derived from simpler irreducible tensors with fewer indices, namely S_{ij} and R, Lanczos [26] thought that the Weyl tensor can also be derivable from a simpler tensor field. It is possible to recreate the Weyl tensor (the gravitational field) through the process of covariant differentiation of a tensor field L_{ijk} (this provides yet another anology between electromagnetism and gravitation). The tensor L_{ijk} is now known as Lanczos potential or Lanczos spin-tensor and satisfies the following symmetries

$$\text{(40 conditions)} \quad L_{ijk} = -L_{jik} \tag{11}$$

$$\text{(4 conditions)} \quad L^{t}_{it} = 0 \quad (\text{or, } g^{kl}L_{kil} = 0) \tag{12}$$

$$\text{(4 conditions)} \quad L_{ijk} + L_{jki} + L_{kij} = 0 \quad (\text{or, } {}^{*}L^{t}_{it} = 0) \tag{13}$$

In this way the tensor field L_{ijk}, which has atmost sixtyfour independent components, has been reduced to atmost sixteen independent components. In order to have a perfect match with the Weyl tensor, Lanczos imposed six equations

$$L^{k}_{ij;k} = 0 \tag{14}$$

so that L_{ijk} is a field with only ten effective degrees of freedom. Equation (14) is known as Lanczos differential guage condition.

Lanczos [26] originally created the gravitational field through the equation

$$C_{hijk} = L_{[hi][j;k]} + L_{[jk][h;i]} - {}^{*}L^{*}{}_{[hi][j;k]} - {}^{*}L^{*}{}_{[jk][h;i]} \tag{15}$$

where dual operation is applied to each pair of antisymmetric brackets as indicated and the double dual is defined as ${}^{*}A^{*}{}_{hijk} = \frac{1}{4}\eta_{hilm}\eta_{jkno}A^{lmno}$.

Equation (15) is now known as Weyl-Lanczos equation and has been written by Dolan and Kim [13] in a more convenient form as

$$\begin{aligned} C_{hijk} &= L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} + L_{(hk)}g_{ij} + L_{(ij)}g_{hk} - L_{(hj)}g_{ik} - L_{(ik)}g_{hj} \\ &\quad + \frac{2}{3}L^{pq}{}_{p;q}(g_{hj}g_{ik} - g_{hk}g_{ij}) \end{aligned} \tag{16}$$

where

$$L_{hk} = L^{t}{}_{hk;t} - L^{t}{}_{ht;k} \tag{17}$$

and round bracket denotes symmetrization.

From (12), (14) and (17), the Weyl-Lanczos equation (16) can also be expressed as

$$\begin{aligned} C_{hijk} &= L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} + \frac{1}{2}\big(L^{p}{}_{hk;p} + L^{p}{}_{kh;p}\big)g_{ij} + \frac{1}{2}\big(L^{p}{}_{ij;p} + L^{p}{}_{ji;p}\big)g_{hk} \\ &\quad - \frac{1}{2}\big(L^{p}{}_{hj;p} + L^{p}{}_{jh;p}\big)g_{ik} - \frac{1}{2}\big(L^{p}{}_{ik;p} + L^{p}{}_{ki;p}\big)g_{hj} \end{aligned} \tag{18}$$

Although the existence of a tensor L_{ijk} as a potential to the Weyl tensor C_{hijk} was established by Lanczos in 1962, there was a little development in the subject for quite sometime (for the state of affairs upto 2000, see [4]). In recent times the Lanczos potential has attracted the attention of a number of authors [2, 6–21, 24–26, 28–33]. For a given geometry the construction of Lanczos potential L_{ijk} is equivalent to solving (16)/(18) with (12)–(14) as constraints. There are several methods of solving (16)/(18) although none of them are straight forward as one would like them to be (cf., [1, 4, 9, 12, 14, 21, 22, 27]). However, in this paper, a yet another method for solving Weyl-Lanczos equation has been given and thus a solution of Weyl Lanczos equation has been obtained for arbitrary Petrov type D vacuum spacetimes using GHP-formalism.

2 NP-Formalism and Weyl-Lanczos Equations

The Newman-Penrose (NP) formalism [28] is a tetrad formalism with a special choice of basis vectors. The beauty of this formalism, when it was first proposed by Newman and Penrose in 1962, was precisely in their choice of a null basis which was customary till then. The underlying motivation for a choice of null basis was Penrose's strong belief that the essential elements of a spacetime is its light cone structure which makes possible the introduction of a spinor basis. The expanded system of equations connecting the spinor components of the Riemann tensor with the components of the spinor connections (spin coefficients) has become known as the system of NP equations. It is possible that the formalism may look somewhat cumbersome with long formulas and tedious calculations, and usually creats some psychological barrier in handling and using NP-method; but once the initial hurdle is crossed, the formalism offers deep insight into the symmetries of the spacetime.

NP formalism has been proved fruitful in the past for finding the exact solutions of the Einstein field equation, asymptotic behavior of the gravitational field and symmetries of the spacetime etc (cf., [2]). This formalism has also been used by a number of authors [8–10, 12, 14, 17, 27, 34] for constructing the Lanczos potential of some special metrics.

In this section, we shall find the NP-version of Weyl-Lanczos equations (18). These results offer corrections to some typographical errors that has occurred in the earlier NP versions of Weyl-Lanczos equations (cf., [10, 34]).

Let

$$Z^a{}_\mu = \{l^a, n^a, m^a, \bar{m}^a\} \tag{19}$$

be the complex null tetrad ($\mu = 1, 2, 3, 4$), where l^a, n^a are real vectors and $m^a, \bar{m}^a$ are the complex null vectors. All the inner products between the tetrad vectors vanish except $l^a n_a = 1 = -m^a \bar{m}^a$.

From the definition of the differential operators

$$D \equiv l^a \nabla_a, \qquad \triangle \equiv n^a \nabla_a, \qquad \delta \equiv m^a \nabla_a, \qquad \bar{\delta} \equiv \bar{m}^a \nabla_a \tag{20}$$

it is possible to write

$$\nabla_a = l_a \triangle + n_a D - \bar{m}_a \delta - m_a \bar{\delta} \tag{21}$$

It is now easy (although lengthy) to see that when the tetrad (19) with (20) and (21) is projected on (18), we have the following set of five equations, which are the NP version of the Weyl-Lanczos equation (18):

$$\Psi_0 = 2[(-\delta + \bar{\alpha} + 3\beta)L_0 + (D - 3\epsilon + \bar{\epsilon})L_4 - \bar{\pi} L_0 - 3\sigma L_1 - \bar{\rho} L_4 + 3\kappa L_5] \tag{22}$$

$$\begin{aligned}\Psi_1 = {} & (-\triangle + 3\gamma + \bar{\gamma})L_0 + (-\delta + \bar{\alpha} + \beta)L_1 + (\bar{\delta} - 3\alpha + \bar{\beta})L_4 + (D + \bar{\epsilon} - \epsilon)L_5 \\ & + (\mu - \bar{\mu})L_0 - (\tau + \bar{\pi})L_1 - 2\sigma L_2 - (\bar{\tau} + \pi)L_4 + (3\rho - \bar{\rho})L_5 + 2\kappa L_6\end{aligned} \tag{23}$$

$$\begin{aligned}\Psi_2 = {} & (-\triangle + \gamma + \bar{\gamma})L_1 + (-\delta + \bar{\alpha} - \beta)L_2 + (\bar{\delta} - \alpha + \bar{\beta})L_5 + (D + \bar{\epsilon} + \epsilon)L_6 - \nu L_0 \\ & + (2\mu - \bar{\mu})L_1 - (2\tau + \bar{\pi})L_2 - \sigma L_3 - \lambda L_4 - (\bar{\tau} + 2\pi)L_5 + (2\rho - \bar{\rho})L_6 + \kappa L_7\end{aligned} \tag{24}$$

$$\begin{aligned}\Psi_3 = {} & (-\triangle - \gamma + \bar{\gamma})L_2 + (-\delta + \bar{\alpha} - 3\beta)L_3 + (\bar{\delta} + \alpha + \bar{\beta})L_6 + (D + \bar{\epsilon} + 3\epsilon)L_7 \\ & + 2\nu L_1 + (3\mu - \bar{\mu})L_2 - (\tau + \bar{\pi})L_3 - 2\lambda L_5 - (\bar{\tau} + 3\pi)L_6 + (\rho - \bar{\rho})L_7\end{aligned} \tag{25}$$

$$\Psi_4 = 2[(-\triangle - 3\gamma + \bar{\gamma})L_3 + (\bar{\delta} + 3\alpha + \bar{\beta})L_7 + 3\nu L_2 - \bar{\mu} L_3 - 3\lambda L_6 - \bar{\tau} L_7] \tag{26}$$

where

$$\begin{aligned}\Psi_0 &= C_{hijk} l^h m^i l^j m^k \\ \Psi_1 &= C_{hijk} l^h m^i l^j n^k \\ \Psi_2 &= C_{hijk} l^h m^i \bar{m}^j n^k \\ \Psi_3 &= C_{hijk} l^h n^i \bar{m}^j n^k \\ \Psi_4 &= C_{hijk} \bar{m}^h n^i \bar{m}^j n^k\end{aligned} \tag{27}$$

are the five complex Weyl scalars and

$$\begin{aligned} L_0 &= L_{hij} l^h m^i l^j, & L_1 &= L_{hij} l^h m^i \bar{m}^j \\ L_2 &= L_{hij} \bar{m}^h n^i l^j, & L_3 &= L_{hij} \bar{m}^h n^i \bar{m}^j \\ L_4 &= L_{hij} l^h m^i m^j, & L_5 &= L_{hij} l^h m^i n^j \\ L_6 &= L_{hij} \bar{m}^h n^i m^j, & L_7 &= L_{hij} \bar{m}^h n^i n^j \end{aligned} \tag{28}$$

are the eight complex Lanczos scalars.

Using 18 NP equations [28] it is easy to solve the set of equation (22)–(26) as 18 NP equations provide a method for integrating Weyl-Lanczos equations (18).

3 GHP-Formalism

A slightly less well known tetrad formalism is the formalism given by Geroch, Held and Penrose [23]. This formalism is called the compacted spin-coefficient formalism or GHP-formalism. It is more concise and efficient than the NP-formalism and deals only with the quantities that transform properly under those Lorentz transformations which leave invariant the two null directions, that is, under boost in l–n plane and under rotation in the m–$\bar{m}$ plane perpendicular to these two directions. In this section, we shall present a brief review of GHP-formalism that is needed for the subsequent investigations. The details may be seen in [1] and [23].

The GHP-formalism deals with the scalars associated with a tetrad $\{l^i, n^i, m^i, \bar{m}^i\}$ where the scalars undergo transformation

$$\eta \longrightarrow \lambda^p \bar{\lambda}^q \eta \tag{29}$$

whenever the tetrad is changed according to

$$\begin{aligned} l^i &\longrightarrow \lambda\bar{\lambda} l^i, & n^i &\longrightarrow \lambda^{-1}\bar{\lambda}^{-1} n^i \\ m^i &\longrightarrow \lambda\bar{\lambda}^{-1} m^i, & \bar{m}^i &\longrightarrow \lambda^{-1}\bar{\lambda} \bar{m}^i \end{aligned} \tag{30}$$

Such a scalar is called a spin and boost weighted scalar of type $\{p, q\}$. It may be noted that $l^i, n^i, m^i, \bar{m}^i$ may be regarded as vectors of type $\{1, 1\}, \{-1, -1\}, \{1, -1\}, \{-1, 1\}$, respectively. Here we shall make use of 'prime' systematically to denote the operation of effecting the replacement:

$$(l^i)' = n^i, \qquad (n^i)' = l^i, \qquad (m^i)' = \bar{m}^i, \qquad (\bar{m}^i)' = m^i \tag{31}$$

Since the bar and prime commute, we write $\bar{\eta}'$ without ambiguity. Moreover, the prime operation is involuntary upto sign

$$(\eta')' = (-1)^{p+q} \eta \tag{32}$$

The Weyl scalars (27), Lanczos scalars (28) and the spin coefficients have the spin and boost of types as indicated below:

$$\Psi_0 : \{4, 0\}, \qquad \Psi_1 : \{2, 0\}, \qquad \Psi_2 : \{0, 0\}, \qquad \Psi_3 : \{-2, 0\}, \qquad \Psi_4 : \{-4, 0\} \tag{33}$$

$$\begin{aligned} &L_o:\{3,1\}, \qquad L_1:\{1,1\}\\ &L_2:\{-1,1\}, \qquad L_3:\{-3,1\}\\ &L_4:\{3,-1\}, \qquad L_5:\{1,-1\}\\ &L_6:\{-1,-1\}, \qquad L_7:\{-3,-1\} \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\kappa:\{3,1\}, \qquad \sigma:\{3,-1\}, \qquad \rho:\{1,1\}, \qquad \tau:\{1,-1\}\\ &\kappa':\{-3,-1\}, \qquad \sigma':\{-3,1\}, \qquad \rho':\{-1,-1\}, \qquad \tau':\{-1,1\} \end{aligned} \tag{35}$$

where the spin coefficient $\kappa', \sigma', \dots,$ etc. are the spin coefficients defined by Newman Penrose as follows:

$$\nu=-\kappa', \qquad \lambda=-\sigma', \qquad \mu=-\rho', \qquad \pi=-\tau', \qquad \alpha=-\beta', \qquad \gamma=-\epsilon' \tag{36}$$

Out of twelve spin-coefficients $(\kappa,\sigma,\rho,\tau,\kappa',\sigma',\rho',\tau',\beta,\beta',\epsilon,\epsilon')$ only eight given by (35) are of good spin and boost and the remaining four $\beta,\beta',\epsilon,\epsilon'$ appear in the definition of the derivatives so that the derivative may not behave badly under spin and boost transformation. For a scalar η of type $\{p,q\}$, the derivative operators are defined as

$$\begin{aligned} &þ\eta=(D-p\epsilon-q\bar{\epsilon})\eta, \qquad þ'\eta=(D'+p\epsilon'+q\bar{\epsilon}')\eta\\ &ð\eta=(\delta-p\beta+q\bar{\beta}')\eta, \qquad ð'\eta=(\delta'+p\beta'-q\bar{\beta})\eta \end{aligned} \tag{37}$$

where $D'=\triangle$ and $\delta'=\bar{\delta}$.

The types of the derivatives (37) are

$$þ:\{1,1\}, \qquad þ':\{-1,-1\}, \qquad ð:\{1,-1\}, \qquad ð':\{-1,1\} \tag{38}$$

The basic quantities with which we are concerned here are eight spin coefficients $\kappa,\sigma,\rho,\tau,\kappa',\sigma',\rho',\tau'$ and the four differential operators $þ, þ', ð, ð'$. There is the operation of complex conjugation and also we may consider the prime as effectively an allowable operator on the system. We also have

$$\bar{þ}=þ, \qquad \bar{þ}'=þ', \qquad \bar{ð}=ð', \qquad \bar{ð}'=ð \tag{39}$$

$$\overline{þ\eta}=\bar{þ}\bar{\eta}, \qquad \overline{ð\eta}=\bar{ð}\bar{\eta} \tag{40}$$

$$\begin{aligned} &(þ\eta)'=þ'\eta', \qquad (þ'\eta)'=þ\eta'\\ &(ð\eta)'=ð'\eta', \qquad (ð'\eta)'=ð\eta' \end{aligned} \tag{41}$$

As a consequence of above considerations we have

$$\Psi_0=\Psi_4', \qquad \Psi_1=\Psi_3', \qquad \Psi_2=\Psi_2', \qquad \Psi_3=\Psi_1', \qquad \Psi_4=\Psi_0' \tag{42}$$

$$\begin{aligned} &L_4=-L_3', \qquad L_5=-L_2'\\ &L_6=-L_1', \qquad L_7=-L_0' \end{aligned} \tag{43}$$

and 18 NP field equations reduce to the following six field equations

$$\eth\rho - \eth'\sigma = \tau(\rho - \bar{\rho}) + \kappa(\bar{\rho}' - \rho') - \Psi_1 \tag{44a}$$

$$þ\rho - \eth'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \tau'\kappa \tag{44b}$$

$$þ\sigma - \eth\,\kappa = \sigma(\rho + \bar{\rho}) - \kappa(\tau + \bar{\tau}') + \Psi_0 \tag{44c}$$

$$þ\tau - þ'\kappa = \rho(\tau - \bar{\tau}') + \sigma(\bar{\tau} - \tau') + \Psi_1 \tag{44d}$$

$$\eth\tau - þ'\sigma = -\rho'\sigma - \bar{\sigma}'\rho + \tau^2 + \kappa\bar{\kappa}' \tag{44e}$$

$$þ'\rho - \eth'\tau = \rho\bar{\rho}' + \sigma\sigma' - \tau\bar{\tau} - \kappa\kappa' - \Psi_2 \tag{44f}$$

When the prime operation is applied to each of these equation we get six more equations.

Therefore when we work with GHP formalism, we have to deal only with κ, σ, ρ, τ, Ψ_0, Ψ_1, Ψ_2, $þ$, $\eth$, L_0, L_1, L_2, L_3 and the six field equations (44). The remaining are the primed versions of these. In this way GHP-formalism allows a considerable amount of simplification as compared to that of NP-formalism.

4 Type D Spacetimes and Lanczos Potential

The study of Petrov type D gravitational field is an important activity as most of the physically significant metrics are of Petrov type D. Some of the familiar members of this class are Schwarzchild, Riessner-Nördstrom, Kerr, Kerr-Newman and Gödel metrics. Moreover, since GHP-formalism has proved useful in the past for studying the Petrov type D gravitational fields [5, 24], it therefore seems worthwhile to have a study of Lanczos potential for such gravitational fields through GHP-formalism. In this section we have obtained GHP-versions of Weyl-Lanczos equations (18) and differential gauge conditions (14). These results alongwith GHP-field equations (44) and their primed versions are used to obtain the Lanczos potential for an arbitrary Petrov type D spacetime; which in turn provides a solution to Weyl-Lanczos equations.

Now using (36), (37), (42) and (43) in Weyl-Lanczos equations (22)–(26), we have the following set of three coupled linear differential equations:

GHP version of Weyl-Lanczos equations

$$\Psi_0 = -2þL_3' - 2\eth L_0 + 2\bar{\tau}'L_0 - 6\sigma L_1 - 6\kappa L_2' + 2\bar{\rho}L_3' \tag{45}$$

$$\begin{aligned}\Psi_1 = {} & -þ'L_0 - \eth L_1 - þL_2' - \eth'L_3' + (\bar{\rho}' - \bar{\rho})L_0 - (\bar{\tau} + \bar{\tau}')L_1 \\ & - 2\kappa L_1' - 2\sigma L_2 - (3\rho - \bar{\rho})L_2' - (\tau' - \bar{\tau})L_3'\end{aligned} \tag{46}$$

$$\begin{aligned}\Psi_2 = {} & -þ'L_1 - þL_1' - \eth L_2 - \eth'L_2' - \kappa L_0' + \kappa'L_0 + (\bar{\rho}' - 2\rho')L_1 + \rho L_1' \\ & + (\bar{\tau}' - 2\tau)L_2 + (\bar{\tau} - 2\tau' - 2\rho)L_2' - \sigma L_3 - \sigma'L_3'\end{aligned} \tag{47}$$

Since $\Psi'_0 = \Psi_4$ and $\Psi'_1 = \Psi_3$ [cf., (42)]; the remaining two Weyl-Lanczos equations are the primed version of (45) and (46).

The GHP version of differential guage condition (14) is given by the following set of three equations:

$$þ'L_2 + \eth L_3 + \eth'L_1' - þL_0' + 2\kappa'L_1 - (\rho' + \bar{\rho}')L_2 + (\tau - \bar{\tau}')L_3$$

$$+2\sigma' L'_2 - (\bar{\tau} - 3\tau')L'_1 + (\bar{\rho} + \rho)L'_0 = 0 \tag{48}$$

$$\begin{aligned} &\text{þ}' L_0 + \eth L_1 + \eth' L'_3 - \text{þ} L'_2 - (\rho' + \bar{\rho}')L_0 + (3\tau + \bar{\tau}')L_1 - 2\sigma L_2 \\ &- (\bar{\tau} + \tau')L'_3 + (\bar{\rho} + 3\rho)L'_2 - 2\kappa L'_1 = 0 \end{aligned} \tag{49}$$

$$\begin{aligned} &\text{þ}' L_1 - \eth L_2 + \eth' L'_2 + \text{þ} L'_1 + \kappa' L_0 - (2\rho' + \bar{\rho}')L_1 + (2\tau - \bar{\tau}')L_2 \\ &- \sigma L_3 + \sigma' L'_3 - (\bar{\tau} + 2\tau')L'_2 - (\bar{\rho} + 2\rho)L'_1 - \kappa L'_0 = 0 \end{aligned} \tag{50}$$

By comparing (34) and (35) one can observe that there is a proportionality between Lanczos scalars and spin coefficients, for example, $L_0 \propto \kappa$, $L_1 \propto \rho$, $L_2 \propto \tau'$, $L_3 \propto \sigma'$, etc, that is, there is some linear relationship between Lanczos scalars and spin coefficients. This is precisely the case with some of the important metrics (cf., [17, 19–22]). Now consider an arbitrary vaccum spacetime of Petrov type D with the null tetrad $\{l^i, n^i, m^i, \bar{m}^i\}$ or $\{l^i, l'^i, m^i, \bar{m}^i\}$. Choose l^i and n^i to lie in the direction of the degenerate principal null vectors so that $\Psi_0 = \Psi_1 = \Psi'_1 = \Psi'_0$ and $\Psi_2 \neq 0$. Thus, using Weyl-Lanczos equations (45)–(47), the primed version of (45) and (46) and GHP-field equations (44a)–(44f), a possible general solution is given by

$$\begin{aligned} L_0 &= \kappa, & L_4 &= -L'_3 = -\sigma \\ L_1 &= \tfrac{1}{3}\rho, & L_5 &= -L'_2 = -\frac{1}{3}\tau \\ L_2 &= -\frac{1}{3}\tau', & L_6 &= -L'_1 = \tfrac{1}{3}\rho' \\ L_3 &= -\sigma', & L_7 &= -L'_0 = \kappa' \end{aligned} \tag{51}$$

Hence, once (51) are known, it may be noted from the completeness relation

$$L_{ijk} = K_{ijk} + \bar{K}_{ijk} \tag{52}$$

between the Lanczos tensor L_{ijk} and Lanczos scalars L_r $(r = 0, 1, \ldots, 7)$ that we can construct the Lanczos spin tensor which in turn generates the Weyl tensor (gravitational field) through Weyl-Lanczos equations (18) where

$$\begin{aligned} K_{ijk} = {} & L_0 U_{ij} n_k + L_1(M_{ij} n_k - U_{ij} m_k) + L_2(V_{ij} n_k - M_{ij} m_k) - L_3 V_{ij} m_k \\ & + L'_3 U_{ij} \bar{m}_k - L'_2(U_{ij} l_k - M_{ij} \bar{m}_k) - L'_1(M_{ij} l_k - V_{ij} \bar{m}_k) - L'_0 V_{ij} l_k \end{aligned} \tag{53}$$

and

$$\begin{aligned} M_{ij} &= l_i n_j - l_j n_i + m_i \bar{m}_j - m_j \bar{m}_i \\ U_{ij} &= -n_i \bar{m}_j + n_j \bar{m}_i \text{ and } V_{ij} = l_i m_j - l_j m_i \end{aligned} \tag{54}$$

Moreover, since the spacetime under consideration is of Petrov type D, the Goldberg-Sachs theorem in such case demands that

$$\kappa = \sigma = \kappa' = \sigma' = 0 \tag{55}$$

and therefore (51) leads to a considerable simpler form of Lanczos scalars for Petrov type D fields as

$$\begin{aligned} L_0 &= 0, & L_4 &= -L'_3 = 0 \\ L_1 &= \frac{1}{3}\rho, & L_5 &= -L'_2 = -\frac{1}{3}\tau \\ L_2 &= -\frac{1}{3}\tau', & L_6 &= -L'_1 = \frac{1}{3}\rho' \\ L_3 &= 0, & L_7 &= -L'_0 = 0 \end{aligned} \tag{56}$$

Example Consider the Kerr spacetime for which the only non-vanishing spin-coefficients are ρ, ρ', τ and τ' and the non-zero component of Weyl scalar is Ψ_2. Since for Kerr spacetime $\kappa = \sigma = \kappa' = \sigma' = 0$; $\Psi_0 = \Psi_1 = \Psi'_1 = \Psi'_0 = 0$ and $\Psi_2 \neq 0$, therefore using (56), GHP field equations (44) and GHP Bianchi identities (A.1)–(A.4) along with their primed versions, we have

$$\tau = -\psi\rho\bar{\rho} \tag{57}$$

$$\tau' = \psi\rho^2 \tag{58}$$

$$þ\tau' = ð'\rho = 2\rho\tau' \tag{59}$$

$$\Psi_2 = M\rho^3 \tag{60}$$

where M is the mass parameter of the Kerr spacetime and the constant of integration ψ satisfies $þ\psi = 0$ [5, 12].

From (56), (58) and (60), the Lanczos potential for the Kerr spacetime is given by

$$L_1 = \frac{1}{3}\left(\frac{\Psi_2}{M}\right)^{\frac{1}{3}}, \qquad L_2 = -\frac{\psi}{27}\left(\frac{\Psi_2}{M}\right)^{\frac{2}{3}} \tag{61}$$

along with their primed versions. Equation (61) clearly shows that the Lanczos potential of the Kerr spacetime is related to the mass parameter of the Kerr spacetime and the Coulomb component of the gravitational field. Moreover, the Lanczos potential of the Kerr black hole depends upon only one of the spin coefficient ρ or τ' (as they are related through (58)).

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Appendix

GHP Bianchi Identities

$$þ\Psi_1 - ð'\Psi_0 = -\tau'\Psi_0 + 4\rho\Psi_1 - 3\kappa\Psi_2 \tag{A.1}$$

$$þ\Psi_2 - ð'\Psi_1 = -\sigma'\Psi_0 - 2\tau'\Psi_1 + 3\rho\Psi_2 - 3\kappa\Psi'_1 \tag{A.2}$$

$$þ\Psi'_0 - ð'\Psi'_1 = 3\sigma'\Psi_2 - 4\tau'\Psi'_1 + \rho\Psi'_0 \tag{A.3}$$

$$þ\Psi'_1 - ð'\Psi_2 = 2\sigma'\Psi_1 - 3\tau'\Psi_2 + 2\rho\Psi'_1 - \kappa\Psi'_0 \tag{A.4}$$

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